REMARKS ON J.W. REYN'S PAPER "DIFFERENTIAL-GEOMETRIC CONSIDERATIONS ON THE HODOGRAPH TRANSFORMATION FOR IRROTATIONAL CONICAL FLOW"

(ZAMECHANIE K STAT'E D.V. REINA "DIFFERENTSIAL'NO-GEOMETRICHESKIE RASSMOTRENIIA PREOBRAZOVANIIA GODOGRAFA DLIA KONICHESKOGO TECHENIIA")

PMM Vol. 26, No. 4, 1962, pp. 793-797

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(Received November 1, 1961)

In Reyn's paper [1], methods of differential geometry were employed to study the properties of the integral surfaces of Busemann's equation [2], which describes the irrotational conical flow of a gas.

As an example of the application of these methods, he considered the problem of the supersonic flow of an inviscid gas past a triangular plate at an angle of attack, without any side-slip. Reyn presented his proof of the impossibility of continuous flow on the upper surface of the plate, and some changes in the pattern of the flow, neither of which is sufficiently justified.

The above point, together with the desire to reply to some broad critical remarks of Reyn on papers [3-7], compelled the author to write the present note.

Let us consider a triangular plate having an angle of attack and no side-slip in an inviscid gas flow, having a speed W_1 and a Mach number $M_1 > 1$. We assume that the edges of the wing are supersonic; thus the conical flows on the top and on the bottom of the wing do not interact, and may be treated separately (Fig. 1).

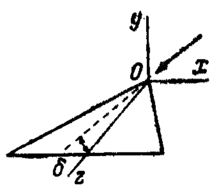

Fig. 1.

In conical flows, the Cartesian components of the velocity, u, v, w, the entropy S, and the pressure p all depend only on angular coordinates, for which we set $\xi = x/z$, $\eta = y/z$, the z-axis being directed along the axis of symmetry of the wing (Fig. 1). For irrotational conical flows, the conical potential $F(\xi, \eta) = z^{-1}\varphi(x, y, z)$ (φ being the velocity potential) satisfies the equation

$$L(F) = AF_{\xi\xi} + 2BF_{\xi\eta} + CF_{\eta\eta} = 0 \qquad (1)$$

$$A = a^2(1+\xi^2) - (u - \xi w)^2$$
$$B = a^2\xi\eta - (u - \xi w)(v - \eta w)$$
$$C = a^2(1+\eta^2) - (v - \eta w)^2$$
$$a^2 = a_1^2 - \frac{\gamma - 1}{2}(u^2 + v^2 + w^2 - W_1^2)$$

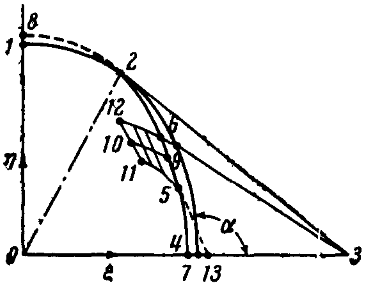

Fig. 2.

Here a_1 is the sound speed in the unperturbed flow, a the local sound speed, and γ the adiabatic exponent. Because of symmetry, we show the mapping of the upper flow [6-8] in the $\xi\eta$-plane only for $\xi > 0$ (Fig. 2).

The wing is represented by the segment *0-3*. The envelope of the Mach cones (of the unperturbed flow) with vertices on the lateral edge is represented by the arc *1-2* of the Mach cone with vertex at the point *0* (Fig. 1) and the segment *2-3*. The flow at the lateral edge consists of a Prandtl-Meyer flow followed by a uniform flow bordering on the wing surface, and represented by the region *3-6-9-5-4-3*. The Prandtl-Meyer flow has a bundle of straight characteristics of Equation (1) passing through the point *3* (straight lines *3-2, 3-6*), i.e. it is a centered simple wave. The region of general conical flow above the middle part of the wing is bounded by the shock wave *2-7* (which lies close to the curved characteristic *2-6* of the Prandtl-Meyer flow, the straight characteristic *6-5*, and the arc *5-4* of the Mach cone for the uniform flow in the region *3-4-5-9-6-3*) and also by the arc *1-2* of the Mach cone for the unperturbed flow.

In [8], the author rejected the introduction of "possible" shocks similar to the shock *2-8* near the Mach cone *1-2*, independently of the arguments of Reyn, who writes [1] that the flow must be expanding near the Mach cone *1-2*, and consequently, one should not expect the shock *2-8*. The argument for introducing the shock *2-8* lies not in the fact (which the author did not notice) that the flow must be expanding in the neighborhood of *1-2*, but in the mathematical difficulties in joining the flow near *1-2* to the rest of the flow, particularly at the point *2*, where the flow behind *1-2* must be joined along the streamline *2-0* with the flow behind the shock wave *2-7* or characteristic *2-6*.

Recently, the author [8] found the singular points, justifying the assumption that the flows are joined at point *2*; this permits not introducing the shock *2-8* (this conclusion also applies to other similar instances, for example, to the flow below a triangular plate, the edge of a rectangular plate, etc.). We note that the flow scheme with a shock *2-8* includes the flow scheme with an arc of the Mach cone *1-2* as a special case. Reyn does not consider the question of joining the flows.

If we assume that the flow about the upper part of the plate is continuous, i.e. that the shock *2-7* is absent, then the boundary of the general conical flow will be the line *1-2-6-5-4*. The arc *5-4* of the Mach cone must be included in the boundary for all angles of attack for which the straight characteristic *6-5* extended to the wing surface makes an angle $\alpha > \pi/2$ with it (Fig. 2), since the data on the characteristic *5-13* together with the flow boundary condition on the segment *4-13* uniquely determine the flow in region *5-13-4*, and this flow is uniform. If for increased angles of attack, α becomes less than $\pi/2$, then the arc *5-4* naturally disappears.

The impossibility of a continuous flow was established by the author [4,6] in the following manner. The characteristic *5-6* is straight; hence the flow joining it is a simple wave, bounded by the curvilinear characteristics *6-12* and *5-11*. Let us move along an arbitrary curvilinear characteristic *9-10* of the simple wave from point *9* to point *10*. In the terminology of [4], this movement is "in the direction of the velocity". In this connection [4], if on this characteristic a parabolic point ($AC - B^2 = 0$) of Equation (1) is encountered then it will be a point on the envelope of the straight characteristics of the simple wave; the acceleration on the straight characteristic passing through the parabolic point is everywhere zero, except at the parabolic point, at which the acceleration depends on the direction of approach to the point. This shows that in the region *5-6-12-10-11* of the simple wave, there cannot occur a continuous parabolic line, along which the simple wave may join to an elliptic-type flow occurring inside the region *0-4-5-6-2-1*. (At the point *0* the flow is always elliptic, $AC - B^2 > 0$.) One can consider the case, when the characteristics *6-12* and *5-11* meet at one parabolic singular point. This case is too artificial, as is the other case when a uniform flow region adjoins the characteristic *12-11*. For these reasons, the author already introduced the shock *2-7* in [4].

Reyn uses the following argument. Let us move along the streamline *3-4-0* along the wing from point *3* to the point *4* on the Mach cone *4-5*. After *5-4* the flow must immediately start compressing, but such flows do not exist. By this argument, when passing from the hodograph space to the physical $\xi\eta$-plane the line *5-4* becomes a limiting line (of the second type in the terminology of Reyn), and the flow must change its direction without intersecting *5-4*. All the streamlines must hit the point *0*, where a Ferri singularity occurs, so the flow must yet again change its direction, i.e. there must be another limiting line (of the first type), after which the streamlines reach *0*. Such a situation obtains for all the streamlines.

Consequently, Reyn's argument is based on the fact that the Mach cone *5-4* cannot adjoin a flow which starts as an expansion and then becomes a compression.

This argument is very plausible: however, the flow adjoining *5-4* is a three-dimensional nonaxisymmetric flow, the properties of which are little understood, and therefore the author preferred the argument advanced above to prove the impossibility of continuous flow, since it requires fewer essential assumptions on the properties of the flow.

Assuming the existence of limiting lines, Reyn attempted to find some confirmation of their existence; for this he considered the differential properties of the integral surface in the hodograph space along the characteristic *2-6*. Here Reyn erred. He arrived at the conclusion that at some intermediate point on the characteristic *2-6*, a limiting line of the first kind is tangent to the characteristic, and here a singular point is formed (the acceleration becomes infinite).

We shall show that such singular points cannot occur in the solution adjoining the simple wave along *2-6*. Let us introduce [4] in the neighborhood of the characteristic *2-6* curvilinear coordinates $\varphi = \varphi(\xi, \eta)$, $\psi = \psi(\xi, \eta)$, such that $\varphi = 0$ corresponds to the characteristic *2-6*. Then Equation (1) is transformed into the form

$$F_{\varphi\varphi}Q(\varphi, \varphi) + 2F_{\varphi\psi}Q(\varphi, \psi) + F_{\psi\psi}Q(\psi, \psi) + F_{\varphi}L(\varphi) + F_{\psi}L(\psi) = 0 \qquad (2)$$

$$Q(\varphi, \varphi) = A\varphi_\xi^2 + 2B\varphi_\xi\varphi_\eta + C\varphi_\eta^2, \quad Q(\varphi, \psi) = A\varphi_\xi\psi_\xi + B(\varphi_\xi\psi_\eta + \varphi_\eta\psi_\xi) + C\varphi_\eta\psi_\eta \text{ etc.}$$

Let f_2, f_1 be the limit values taken while the curve $\varphi = 0$ is approached from different sides; let $[f] = f_2 - f_1$ denote the jump in the function f.

The functions F, F_ξ, F_η are continuous across $\varphi = 0$; it may be shown that

$$[F_{\xi\xi}] = \lambda\varphi_\xi^2, \qquad [F_{\xi\eta}] = \lambda\varphi_\xi\varphi_\eta, \qquad [F_{\eta\eta}] = \lambda\varphi_\eta^2$$

The coefficient of the jump in the second derivatives $\lambda = [F_{\varphi\varphi}]$ satisfies the equation

$$2Q(\varphi, \psi)\frac{d\lambda}{d\psi} + \lambda\{Q(\varphi, \varphi)_{F_\varphi}\lambda + Q_1(\varphi, \varphi)_\varphi + Q(\varphi, \varphi)_{F_\varphi}(F_{\varphi\varphi})_1 +$$
$$+ 2F_{\varphi\psi}Q(\psi, \psi)_{F_\varphi} + F_{\psi\psi}Q(\psi, \psi)_{F_\varphi} + F_\varphi L(\varphi)_{F_\varphi} + F_\psi L(\psi)_{F_\varphi} + L(\varphi)\} = 0 \qquad (3)$$

Subscript 1 denotes the values of $Q(\varphi, \varphi)_\varphi$ and $F_{\varphi\varphi}$, taken when $\varphi = 0$ is approached from the side of the simple wave. In the simple wave, F has no singular points (except the point 3), hence the coefficients of (3) are regular on the characteristic 2-6. If at some point on 2-6, the acceleration becomes infinite, as was supposed by Reyn, then this point is a singularity of Equation (3). Equation (3) is nonlinear, therefore, its solution may have fixed and moving singularities. A fixed singular point occurs at point 2, where

$$Q(\varphi, \psi) = 0$$

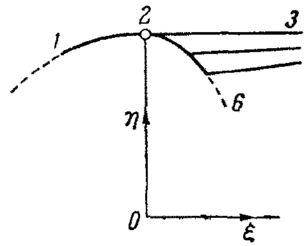

Fig. 3.

There are no other fixed singularities on the characteristic 2-6. In fact, if we move along the curvilinear characteristic 2-6 of the simple wave from point 2 to point 6, i.e. move "in the direction of the velocity" [4], and meet a point where $Q(\varphi_1 \psi) = 0$, then this point is a parabolic point ($AC - B^2 = 0$) for Equation (1). But this point is a point on the envelope of the straight characteristics of the simple wave [4], which is impossible, since this envelope is the single point 3 (Fig. 2).

Let us investigate in detail the behavior of the solution of (3) in the neighborhood of point 2 (Fig. 2). If we rotate the coordinate system, so that z becomes the direction of the velocity on the characteristic 2-3, and so that the characteristic 2-3 is parallel to the ξ-axis (Fig. 3), then the coordinates of point 2 will be $\xi = 0$, $\eta = \eta_1 = (M_1{}^2 - 1)^{-1/2}$, and the equation of the characteristic 2-6 near the point 2 will have the form [4]

$$\eta = \eta_1 - \eta_1{}^{-1/2}\xi^2 + O(\xi^3)$$

Moreover, the derivatives of the conical potential F at the point 2 are

$$F_\xi = F_\eta = F_{\xi\xi} = F_{\xi\eta} = 0, \qquad F_{\eta\eta} = \frac{2W_1}{\gamma + 1} \frac{(M_1{}^2 - 1)^2}{M_1{}^4}$$

Introducing the coordinates φ, ψ, thus defined

$$\varphi = \eta - \eta_1 + \eta_1{}^{-1/2}\xi^2 + O(\xi^3), \qquad \psi = \xi$$

and considering the behavior of F in the simple wave, we may write Equation (3) near the point 2 in the form

$$(\alpha_0\psi + \ldots)\frac{d\lambda}{d\psi} + \lambda\{(\gamma_0 + \ldots) + \lambda(\beta_0 + \ldots)\} = 0$$

where

$$\alpha_0 = 2a_1{}^2(M_1{}^2 - 1)^{1/2}, \qquad \beta_0 = \frac{(\gamma + 1) M_1{}^2 W_1}{(M_1{}^2 - 1)^{3/2}}, \qquad \gamma_0 = 4a_1{}^2(M_1{}^2 - 1)^{1/2}$$

Since $\beta_0 \neq 0$, $\gamma_0/\alpha_0 = 2 > 0$, then [4]

$$\lim_{\psi\to 0}\lambda = -\frac{4W_1}{\gamma+1}\frac{(M_1^2-1)^2}{M_1^4} = -\frac{\gamma_0}{\beta_0} \quad \text{for } \lambda \not\equiv 0$$

Knowing $\lambda(0)$, we find

$$(F_{\eta\eta})_2 = (F_{\eta\eta})_1 + \lambda\varphi_\eta^2 = -\frac{2W_1}{\gamma+1}\frac{(M_1^2-1)^2}{M_1^4} \quad \text{for } \psi\to 0 \tag{4}$$

In fact, either $\lambda \equiv 0$ everywhere on *2-6*, and the quantity $(F_{\varphi\varphi})_2$ coincides with $(F_{\varphi\varphi})_1$ everywhere in the simple wave, or $(F_{\eta\eta})_2$ at the point *2* is given by (4) for any solution adjoining the characteristic *2-6*.

There is a singular point [8] at the point *2* (Fig. 3). When this point is approached from the sides $\xi = +0$ and $\xi = -0$ the singularities are different. We shall not consider the singularity for $\xi = -0$; we only note that the flow must expand near the point *2*. The solution near point *2* for $\xi = +0$ is given by the relations

$$\begin{aligned} u &= \frac{2W_1}{\gamma+1}\frac{(M_1^2-1)^3}{M_1^4}(2X-\sigma X')\xi^3+\ldots & \left(X = X(\sigma),\ X' = \frac{dX}{d\sigma}\right)\\ v &= -\frac{W_1}{\gamma+1}\frac{(M_1^2-1)^{5/2}}{M_1^4}X'\xi^2+\ldots & \left(\sigma = \frac{\eta_1-\eta}{\xi^2}\eta_1\right)\\ w &= W_1+\frac{W_1}{\gamma+1}\frac{(M_1^2-1)^2}{M_1^4}X'\xi^2+\ldots & \left(\eta_1 = (M_1^2-1)^{-1/2}\right)\end{aligned} \tag{5}$$

Moreover, $X = X(\sigma)$ satisfies the equation

$$(X' + 2\sigma - 4\sigma^2)X'' + (10\sigma - 4)X' - 12X = 0 \tag{6}$$

with the boundary conditions

$$X(1) = 1, \qquad X'(1) = 2 \tag{7}$$

(The conditions result from Formula (2.30) of [8] with $\sigma_0 = 1$, which corresponds to the characteristic *2-6*.) The solutions of (6) satisfying (7) fall into two types, differing in the value of $X''(1)$. The first type of solution is represented in the neighborhood of $\sigma = 1$ by the expansion

$$X(\sigma) = 1 + 2(\sigma-1) + (\sigma-1)^2 + c(\sigma-1)^3 + \frac{c(3c-1)}{2}(\sigma-1)^4 + \ldots \tag{8}$$

where c is an arbitrary constant. The unique solution of the second type is

$$X(\sigma) = 1 + 2(\sigma-1) - (\sigma-1)^2 = -2 + 4\sigma - \sigma^2 \tag{9}$$

Substituting (9) into (5), and changing from σ to ξ, η, we have

$$F_\eta = v = -\frac{W_1}{\gamma+1}\frac{(M_1^2-1)^{5/2}}{M_1^4}\{4\xi^2 - 2(\eta_1-\eta)(M_1^2-1)^{-1/2}\} + \ldots \qquad (10)$$

in the entire quadrant neighborhood $\xi > 0$, $\eta_1 - \eta > 0$. From (10) it follows that when point *2* is approached in whatever direction $F_{\eta\eta}$ is defined by (4).

But then approaching point *2* along the axis $\xi = 0$

$$\frac{\partial}{\partial\eta}(u^2+v^2+w^2) = -\frac{2W_1}{(M_1^2-1)^{1/2}}F_{\eta\eta} = \frac{4W_1^2}{\gamma+1}\frac{(M_1^2-1)^{3/2}}{M_1^4} > 0$$

which is incompatible with the requirement of an expanding flow in the neighborhood of point *2* (Figs. 2,3).

For all the solutions (8), we have the derivative at point *2*

$$F_{\eta\eta} = \frac{2W_1}{\gamma+1}\frac{(M_1^2-1)^2}{M_1^4} \qquad (11)$$

All proper solutions F adjoining the characteristic *2-6* have the property that the derivative $F_{\eta\eta}$ is given by (11) when the point *2* is approached along the characteristic *2-6*, thus $\lambda \equiv 0$; that is, on the entire characteristic *2-6*, the accelerations coincide with the accelerations of a Prandtl-Meyer flow, and therefore, cannot be infinite, as Reyn claimed.

The physical basis for the formation of the shock *2-7* (Fig. 2) is the appearance (in the Prandtl-Meyer flow at the lateral edge) of a velocity component directed at the plane of symmetry of the flow ($\xi = 0$). For this reason, it is natural to assume that the shock *2-7* starts at the point *2*, after which such a component appears on the characteristic *2-6*.

The author also considered a possible variation, when the lateral shock *2-7* is partially located inside the region *0-1-2-6-5-4*, and the shock enters this region at some point on the characteristic *6-5*. An alternate version of Reyn's, which assumes the shock to begin at some intermediate point on the characteristic *2-6*, is improbable; since, if this were the case, then the point where the shock decays must be a singular point, which must show singularities in some higher derivatives of F taken along the characteristic *2-6*. But as shown above, the first derivative has no singularity on *2-6*; the higher derivatives cannot have them either, since the equation for the coefficients of the discontinuity of the higher derivatives is linear, and cannot have any moving singular points.

The author is grateful to S.F. Fal'kovich for discussions on the problem.

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Translated by C.K.C.